

LOCAL DECAY OF SOLUTIONS OF CONSERVATIVE FIRST ORDER HYPERBOLIC SYSTEMS IN ODD DIMENSIONAL SPACE⁽¹⁾

BY
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ABSTRACT. This paper deals with symmetric hyperbolic systems, $\partial u/\partial t = Lu$, where L is equal to the homogeneous, constant coefficient operator L_0 for $|x| > R$. Under the hypothesis that L has simple null bicharacteristics and these propagate to infinity, local decay of solutions and completeness of the wave operators relating solutions of $\partial u/\partial t = Lu$ and solutions of $\partial u/\partial t = L_0 u$ are established. Results of this type for elliptic L are due to Lax and Phillips. The proof here is based, in part, on a new estimate of the regularity of the L^2 -solutions of the equation $Lu + (i\lambda + \varepsilon)u = g$ for smooth g with support in $|x| \leq R$.

Let L be an operator of the form

$$L = \sum_{j=1}^n A_j(x) \frac{\partial}{\partial x_j} + B(x).$$

The coefficients $A_j(x)$ are $k \times k$ hermitian matrices. They are smooth functions of x on \mathbf{R}^n and take constant values A_j^0 for $|x| > R$. The matrix $B(x)$ is also a smooth function of x and it vanishes for $|x| > R$. $B(x)$ is chosen so that L is formally skew-hermitian. Under these hypotheses the graph closure of L defined on $C_0^\infty(\mathbf{R}^n)$ is a skew-hermitian operator on $L^2(\mathbf{R}^n)$. $C_0^\infty(\mathbf{R}^n)$ and $L^2(\mathbf{R}^n)$ denote the spaces of \mathbf{C}^k -valued smooth functions of compact support and \mathbf{C}^k -valued square-integrable functions respectively. The operator

$$L_0 = \sum_{j=1}^n A_j^0 \frac{\partial}{\partial x_j}$$

is also skew-hermitian on $L^2(\mathbf{R}^n)$. Thus L and L_0 generate unitary groups $U(t)$ and $U_0(t)$ on $L^2(\mathbf{R}^n)$, and the solutions to the initial value problems $u(0) = f_1 \in C_0^\infty(\mathbf{R}^n)$, $\partial u/\partial t = Lu$, and $u(0) = f_2 \in C_0^\infty(\mathbf{R}^n)$, $\partial u/\partial t = L_0 u$, are given by $U(t)f_1$ and $U_0(t)f_2$.

In this paper we are concerned with the local decay of solutions to the hyperbolic system of equations $\partial u/\partial t = Lu$. Defining

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$$\|f\|^r = \left(\int_{|x| < r} |f|^2 dx \right)^{1/2}$$

for $f \in L^2(\mathbf{R}^n)$, we say a solution with initial data f decays locally if $\lim_{t \rightarrow \pm\infty} \|U(t)f\|^r = 0$ for all $r > 0$. From the standpoint of scattering, the desirable result is that local decay holds for all initial data orthogonal to H_0 , the subspace of $L^2(\mathbf{R}^n)$ spanned by the eigenvectors of L .

From the theory of hyperbolic equations one might suspect that local decay would depend on the behavior of the bicharacteristic curves of $\partial/\partial t - L$, or, equivalently, the bicharacteristic curves of L . However, Lax and Phillips [6] have shown if L is elliptic one always has local decay on $L^2(\mathbf{R}^n) \ominus H_0$. The results we present here indicate it is the behavior of the null bicharacteristics of L (these are absent when L is elliptic) which influences the local decay of solutions. Our result is the following

Theorem I. Let $A(x, \xi) = \sum_{j=1}^n A_j(x) \xi_j$, $(x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n$.

Assume the following:

(i) The null space of $A(x, \xi)$ is at most one dimensional for $\xi \neq 0$. This implies there is a smooth function $\tau(x, \xi)$, defined on a conic neighborhood of $S = \{(x, \xi) \mid \det A(x, \xi) = 0, |\xi| = 1\}$, and vanishing on S , such that $\det(A(x, \xi) - \tau(x, \xi)I) = 0$.

(ii) Given $(x_0, \xi_0) \in S \cap \{(x, \xi) \mid |x| \leq R\}$ and $r > 0$ there is a T such that the bicharacteristic curve $(x(t), \xi(t))$, $\dot{x}(t) = -(\partial/\partial \xi)\tau(x, \xi)$, $\dot{\xi}(t) = (\partial/\partial x)\tau(x, \xi)$, $x(0) = x_0$, $\xi(0) = \xi_0$ satisfies $|x(t)| > r$ for $|t| > T$.

(iii) n is odd.

Under these assumptions:

(a) The point spectrum of L has no accumulation points and with the possible exception of zero has finite multiplicity.

(b) Given $f \in L^2(\mathbf{R}^n)$ there exist $g_{\pm} \in L^2(\mathbf{R}^n) \ominus H_0$ such that

$$\lim_{t \rightarrow \pm\infty} \|U(t)g_{\pm} - U_0(t)f\| = 0,$$

and given $g \in L^2(\mathbf{R}^n) \ominus H_0$ there exist $f_{\pm} \in L^2(\mathbf{R}^n)$ such that

$$\lim_{t \rightarrow \pm\infty} \|U(t)g - U_0(t)f_{\pm}\| = 0.$$

(c) For all $f \in L^2(\mathbf{R}^n) \ominus H_0$ and $r > 0$,

$$\lim_{t \rightarrow \pm\infty} \|U(t)f\|^r = 0.$$

One might hope to replace hypothesis (ii) by the assumption that the group velocity $|\partial\tau/\partial\xi|$ does not vanish on S , but Example 4.1 shows this assumption does not imply conclusions (b) or (c) (a slight modification of this example would show it does not imply (a) either). From the standpoint of applications some

weakening of (i) is desirable.⁽²⁾ For instance, the linearized equations of magneto-hydrodynamics fail to satisfy (i). Hypothesis (iii) arises because of our use of the Radon transform in §2 and in the proof of Corollary 1.3.⁽³⁾ In view of recent results of Lax and Phillips [8] it is possible that (iii) is unnecessary. Conclusion (a) is probably much too weak: all known examples (cf. [11]) have no point spectrum except for $\lambda = 0$ and this is an eigenvalue of finite multiplicity. For L elliptic with the unique continuation property this is the general situation (cf. [7, Theorem 3.1]). Under hypotheses (i)–(ii) it is easy to show local decay holds for the group $U_0(t)$ on $L^2(\mathbf{R}^n)$. Hence (c) is an immediate consequence of the second part of (b). Conclusion (b) gives the existence and unitarity of the scattering operator relating $U(t)$ and $U_0(t)$.

Our proof of (b) is modeled on the proof used by Phillips and Sarason [9] to show the existence of the scattering operator in a boundary value problem for an elliptic L with nonelliptic boundary conditions. We construct generalized (not square-integrable) eigenfunctions of L satisfying a type of radiation condition and use these to give an explicit diagonalization of the restriction of L to subspaces H^+ and H^- . The generalized eigenfunctions are constructed by the limiting absorption method. This method was developed rigorously by Eidus [1] and has been widely used in scattering theory to treat elliptic problems and problems of constant deficit (cf. [2] and [10]). To use limiting absorption here we require an estimate which will take the place of the coercive estimates from elliptic theory. This estimate, Theorem 1.1, is proved in §1 and makes essential use of recent work of Hörmander [5]. In §2 we construct the generalized eigenfunctions and the diagonalizations of L on H_+ and H_- . We also prove a crucial preliminary decay result, Theorem 2.2. In §3 we complete the program of [9] by showing $H_+ + H_-$ is dense in $L^2(\mathbf{R}^n) \ominus H_0$. In doing this we make use of an operator-valued function $Q(t)$ which approximately commutes with $\partial/\partial t - L$. The construction of this operator was suggested by the construction in §2.4 of [3].

In the final section, §4, we use $Q(t)$ again to show that the behavior of the nonnull bicharacteristics of L does influence the *uniformity* of the rate of local decay. We show that if L has a simple bicharacteristic which remains in $|x| < R$ as $t \rightarrow \infty$, then for all $t > 0$ and $\epsilon > 0$ one can find initial data f supported in $|x| < R$ such that $\|U(t)f\|^R \geq (1 - \epsilon)\|f\|$. We believe this result could be deduced directly from [4], but the proof given here is short and self-contained.

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⁽²⁾ **Added in proof.** This extension will require methods different from those used here. One can construct a 2×2 operator L in two variables with diagonal top order symbol such that the bicharacteristics of L propagate to infinity and the conclusion of Theorem 1.1 does not hold.

⁽³⁾ **Added in proof.** Patrick Murphy has pointed out that Theorem 3.3 of [7] can be extended to the case of even n , simply by considering an operator in n variables as an operator in $n + 1$ variables with a vanishing coefficient. Hence, Corollary 1.3 (which is conclusion (a)) holds for even n .

1. The coercive estimate. We begin by constructing functions to be used later and fixing notation. Let $S = \{(x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n \mid \det A(x, \xi) = 0, |\xi| = 1\}$, and let $N(x, \xi)$ be the null space of $A(x, \xi)$. By hypothesis $N(x, \xi)$ is one dimensional for $(x, \xi) \in S$. Hence, since $S_R = S \cap \{(x, \xi) \mid |x| \leq R\}$ is compact and $A(x, \xi)$ is independent of x for $|x| \geq R$, there is a $\delta > 0$ such that $A(x, \xi) - zI$ is invertible on $N(x, \xi)^\perp$ for $|z| \leq \delta$, $(x, \xi) \in S$. By continuity $R(x, \xi, z) = (A(x, \xi) - zI)^{-1}$ exists for $|z| = \delta$ for (x, ξ) in an open neighborhood U of S . We may assume $U \cap \{(x, \xi) \mid x = x_0\}$ is independent of x_0 for $|x_0| \geq R$ and each connected component of U intersects S . We define a smooth projection-valued function on U by

$$P(x, \xi) = \frac{-1}{2\pi i} \oint_{|z|=\delta} R(x, \xi, z) dz.$$

For $(x, \xi) \in U$, $A(x, \xi)P(x, \xi) = \tau(x, \xi)P(x, \xi)$, where $\tau(x, \xi)$ is the eigenvalue of $A(x, \xi)$ given by

$$\tau(x, \xi) = \frac{-1}{2\pi i} \oint_{|z|=\delta} z \det R \frac{\partial}{\partial z} (\det R^{-1}) dz.$$

For $(x, \xi) \in S$, $P(x, \xi)$ is the orthogonal projection on $N(x, \xi)$ and $\tau(x, \xi) = 0$. $P(x, \xi)$ and $\tau(x, \xi)$ are extended to a conic neighborhood $V \supset U$ via $P(x, \xi) = P(x, \xi/|\xi|)$ and $\tau(x, \xi) = |\xi|\tau(x, \xi/|\xi|)$.

The chief objective of this section is to prove the following estimate in Sobolev norms ($\|f\|_0^r$ is just the localized L^2 -norm, $\|f\|'$, introduced earlier):

Theorem 1.1. *Let $Lu + (i\lambda \pm \epsilon)u = g$, $g \in C_0^\infty(|x| < R)$, $u \in L^2(\mathbf{R}^n)$, $0 < \epsilon \leq 1$, $|\lambda| \leq K$. Then there is an R' such that given s there is a cutoff $\alpha \in C_0^\infty$, $\alpha(x) = 1$, $|x| \leq R$, for which*

$$\|\alpha u\|_s \leq C_s (\|g\|_s + \|u\|_0^{R'}).$$

R' and C_s is independent of λ , ϵ , and u .

The main ingredient in the proof of Theorem 1.1 is the following estimate due to Hörmander:

Let $p(x, \xi)$ be a real symbol of order one such that $dp \neq 0$ on $p^{-1}(0)$, and let $\gamma(t)$ be a bicharacteristic strip for p contained in $p^{-1}(0)$, such that the projection of $\gamma(t)$ onto $\{(x, \xi) \mid |\xi| = 1\}$ is injective. Then, given t_1, t_2 and conic neighborhoods N_1 and N_2 of $\gamma(t_1)$ and $\{\gamma(t) \mid t_1 \leq t \leq t_2\}$ respectively, there are symbols of order zero, ρ, ρ_1 and ρ_2 , such that ρ_i is supported in N_i , $i = 1, 2$, $\rho = 1$, on a conic neighborhood of $\{\gamma(t) \mid t_1 \leq t \leq t_2\}$ and

$$(1.1) \quad \|\rho(x, D)u\|_s \leq C_{s,k} (\|\rho_1(x, D)u\|_s + \|\rho_2(x, D)p(x, D)u\|_s + \|\rho_2(x, D)u\|_{-k})$$

for any distribution u .

This is Proposition 3.5.1 of [5] restricted to the case of real symbols and restated in quantitative rather than qualitative form (in [5] the conclusion is that ρu is in H^s if $\rho_1 u$ and $\rho_2 p u$ are in H^s , but the proof of this yields the estimate given here). This is an analogue for operators with simple null bicharacteristics of the basic regularity estimates of elliptic theory.

Given a distribution solution u to $Lu + (i\lambda + \epsilon)u = g$ we will prove Theorem 1.1 by using (1.1) together with elliptic estimates. However, this requires us to bound the regularity of u (as measured by $\|\rho_1(x, D)u\|_s$) by a constant multiple of $\|g\|_s + \|u\|_0^{R'}$ at some point $\gamma(t_1)$ on each null bicharacteristic emanating from S_R . By hypothesis (ii), given $(x_0, \xi_0) \in S_R$, there is a T such that the bicharacteristic $(x(t), \xi(t))$ with

$$dx/dt = -(\partial\tau/\partial\xi)(x, \xi), \quad d\xi/dt = (\partial\tau/\partial x)(x, \xi),$$

$x(0) = x_0$, $\xi(0) = \xi_0$, satisfies $|x(t)| > R$ for $|t| > T$. Thus we may always choose $\gamma(t_1) = (x(t_1), \xi(t_1))$ with $|x(t_1)| > R$. Even so, the desired bound does not hold for arbitrary solutions of $Lu + (i\lambda + \epsilon)u = g$. A suitable bound with constants independent of ϵ does hold for the unique solution in $L^2(\mathbf{R}^n)$, as is shown by the following lemma.

Lemma 1.2. *Let $(x(t), \xi(t))$ be the bicharacteristic described above. Then there is a $T_0 > T$ and there are pseudo-differential operators $\varphi_{\pm}(x, D)$ of order zero with symbols $\varphi_{\pm}(x, \xi) = P(x, \xi)$ on conic neighborhoods N_{\pm} of $(x(\pm T_0), \xi(\pm T_0))$ for which the following holds: If u is the square-integrable solution to $Lu + (\lambda \mp i\epsilon)u = g$, $g \in C_0^{\infty}(|x| < R)$, then*

$$\|\varphi_{\pm}(x, D)u\|_s \leq C_s(\|g\|_0 + \|u\|_0^R).$$

Here C_s is independent of ϵ and λ for $0 < \epsilon \leq 1$ and $|\lambda| < K$.

Proof. We assume $Lu + (i\lambda + \epsilon)u = g$. The proof for $Lu + (i\lambda - \epsilon)u = g$ is identical.

Set $(\hat{x}, \hat{\xi}) = (x(-T), \xi(-T))$ and set $P(\xi) = P(x, \xi)$, $\tau(\xi) = \tau(x, \xi)$ for $|x| > R$. By hypothesis (ii), $\partial\tau/\partial\xi(\hat{\xi}) \neq 0$. We rotate coordinates in ξ -space by a matrix 0 , $\xi' = 0\xi$ so that $\xi' = (1, 0, \dots, 0)$ is normal to the surface $\tau(\xi) = 0$ at $\xi = \hat{\xi}$ and $\partial\tau \circ 0^*/\partial\xi'_1(0\hat{\xi}) > 0$. Next we make a second change of coordinates $\eta = \Lambda(\xi') = (\tau(0^*\xi'), \xi'_2, \dots, \xi'_n)$. The jacobian of $\Lambda(\xi')$, $\Lambda'(\xi')$, is homogeneous degree zero and hence $\det \Lambda'(\xi') \neq 0$ on a conic neighborhood of $0\hat{\xi} = \hat{\xi}'$. Suppose there is no conic neighborhood of $\hat{\xi}'$ on which $\Lambda(\xi')$ is injective. Then there are pairs ρ_m^1, ρ_m^2 such that $\rho_m^i/|\rho_m^i| \rightarrow \hat{\xi}'/|\hat{\xi}'|$ as $m \rightarrow \infty$, $i = 1, 2$, and $\Lambda(\rho_m^1) = \Lambda(\rho_m^2)$. Since $\hat{\xi}' \cdot \partial\tau \circ 0^*/\partial\xi'(\hat{\xi}') = 0$, $\hat{\xi}' = (0, \hat{\xi}'_2, \dots, \hat{\xi}'_n) \neq 0$. Hence $\lim_{m \rightarrow \infty} |\rho_m^1|/|\rho_m^2| = 1$, and now $\Lambda(|\hat{\xi}'|\rho_m^1/|\rho_m^1|) = \Lambda(|\hat{\xi}'|\rho_m^2/|\rho_m^1|)$ leads to a contradiction to the injectivity of $\Lambda(\xi')$ on an ordinary neighborhood of $\hat{\xi}'$. Thus, defining $I(\xi) = \Lambda(0\xi)$, $I(\xi)$ is a diffeomorphism of a conic neighborhood C of $\hat{\xi}$ onto a conic neighborhood \tilde{C} of $I(\hat{\xi})$. Let $\varphi(\xi)$ be a symbol of order zero such

that $\varphi(\xi) = 1$ on a conic neighborhood of $\hat{\xi}$ and support φ is contained in C . Define $q(\xi) = \varphi(\xi)P(\xi)$, and let $Q(D)$ be the pseudo-differential operator with symbol $q(\xi)$.

By hypothesis $u \in L^2(\mathbf{R}^n)$, and in the sense of distributions

$$L_0 u + (i\lambda + \varepsilon)u = g + (L_0 - L)u = h.$$

Note support $h \subset |x| \leq R$. Taking fourier transforms and multiplying by $q(\xi)$,

$$i\tau(\xi)q(\xi)\hat{u}(\xi) + (i\lambda + \varepsilon)q(\xi)\hat{u}(\xi) = q(\xi)\hat{h}(\xi).$$

Hence

$$Q(D)u = \mathcal{F}^{-1} \frac{q(\xi)\hat{h}(\xi)}{i(\tau(\xi) + \lambda) + \varepsilon}.$$

Let $\hat{h}_N(\xi) = \psi(|\xi|/N)\hat{h}(\xi)$ and $\hat{u}_N = \psi(|\xi|/N)\hat{u}(\xi)$ where $\psi \in C_0^\infty(\mathbf{R})$ and $\psi(s) = 1$ for $|s| < 1$. Then

$$\begin{aligned} (Q(D)u_N)(x) &= (2\pi)^{-n/2} \int e^{ix \cdot \xi} \frac{q(\xi)\hat{h}_N(\xi)}{i(\tau(\xi) + \lambda) + \varepsilon} d\xi \\ &= (2\pi)^{-n/2} \int \exp(ix \cdot I^{-1}(\eta)) \frac{\hat{H}_N(\eta)}{i(\eta_1 + \lambda) + \varepsilon} d\eta \end{aligned}$$

where $\hat{H}_N(\eta) = q(I^{-1}(\eta))\hat{h}_N(I^{-1}(\eta))(\det(I^{-1})'(\eta))$. Thus

$$(1.2) \quad Q(D)u_N(x) = (2\pi)^{-n} \int \exp(ix \cdot I^{-1}(\eta)) \bar{\psi}(\eta) d\eta \int \exp(-iy \cdot \eta) B_N(y) dy$$

where $\bar{\psi}$ is a symbol of order zero such that $\bar{\psi}(\eta) = 1$ on a conic neighborhood of support $q(I^{-1}(\eta))$ and support $\bar{\psi}$ is in \tilde{C} .

$$(1.3) \quad B_N(y) = \int_{-\infty}^{y_1} \exp((i\lambda + \varepsilon)(s - y_1)) H_N(s, y_2, \dots, y_n) ds$$

and

$$H_N(y) = (2\pi)^{-n/2} \int \exp(iy \cdot I(\xi)) \psi(|\xi|/N) q(\xi) \hat{h}(\xi) d\xi.$$

Let $M = (1/i)|I'(\xi)^*y - x|^{-2}((I'(\xi))^*y - x) \cdot \partial/\partial\xi$ and choose R' so that $|I'(\xi)^*y - x| > 1$ for $|x| \leq R$, $|y| > R'$, $\xi \in \text{support } q$. Then we have for $|y| > R'$ and any integer $r > 0$

$$(1.4) \quad H_N(y) = \left((2\pi)^{-n} \int e^{i(y \cdot I(\xi) - x \cdot \xi)} (M^*)^r \left(\psi\left(\frac{|\xi|}{N}\right) q(\xi) \right) d\xi, h \right).$$

Letting $\varphi \in C_0^\infty(\mathbf{R}^n)$, $\varphi(y) = 1$ for $|y| \leq R'$, (1.4) implies for all m

$$(1.5) \quad \rho_m((1 - \varphi)H_N(y)) \leq c_m \|h\|_{-1}$$

where ρ_m denotes the seminorm

$$\rho_m(f) = \sup_{|\alpha| \leq m} \sum (1 + |x|^2)^m |D^\alpha f|$$

and c_m is independent of N .

Since $\|f\|_{-s} = \sup\{(\nu, f) \mid \nu \in C_0^\infty, \|(1 - \Delta)^{s/2} \nu\|_0 \leq 1\}$, we have (assume s is an even integer)

$$(1.6) \quad \|\varphi H_N\|_{-s} \leq (2\pi)^{-n} \left\| \left(\int e^{i(y \cdot I(\xi) - x \cdot \xi)} \frac{\varphi(y) \psi(|\xi|/N) q(\xi)}{(1 + |I(\xi)|^2)^{s/2}} d\xi, h \right) \right\|_0.$$

As $I(\hat{\xi}) \neq 0$, we may assume that C is chosen so that $I(\xi) \neq 0$ for $\xi \in C \cap \{|\xi| = |\hat{\xi}|\}$. Then (1.6) implies for S sufficiently large

$$(1.7) \quad \|\varphi H_N\|_{-s} \leq c \|h\|_{-1} \quad \text{and } c \text{ is independent of } N.$$

Choose $\rho \in C^\infty(\mathbf{R})$ such that $\rho(x) = 1$ for $x < R'' - 1$ and $\rho(x) = 0$ for $x > -R''$, where support φ is contained in $y_1 > -R''$. Then (1.3) and (1.5) imply for all m

$$(1.8) \quad \rho_m(\rho(y_1) B_N(y)) \leq k_m \|h\|_{-1}$$

where k_m is independent of N , λ and ε .

Using the definition of $\|\cdot\|_{-s}$ we have

$$\begin{aligned} & \| (1 + y_1^2)^{-1} (1 - \rho(y_1)) B_N(y) \|_{-s} \\ &= \sup_{\|(1-\Delta)^{s/2} \nu\|_0 \leq 1} \int \nu(y) (1 + y_1^2)^{-1} \\ & \quad \cdot (1 - \rho(y_1)) \int_{-\infty}^{y_1} e^{(i\lambda + \varepsilon)(s - y_1)} H_N(s, y_2, \dots, y_n) ds dy \\ &= \sup_{\|(1-\Delta)^{s/2} \nu\|_0 \leq 1} \int V(y) H_N(y) dy \end{aligned}$$

where

$$V(y) = \int_{y_1}^{\infty} (1 + s^2)^{-1} (1 - \rho(s)) \exp((i\lambda + \varepsilon)(y_1 - s)) \nu(s, y_2, \dots, y_n) ds.$$

Let $\psi \in C_0^\infty(\mathbf{R}^n)$ equal 1 on support φ . Since

$$\|(1 - \Delta)^{s/2} \psi V\|_0 \leq C \|(1 - \Delta)^{s/2} \nu\|_0,$$

where C is independent of ε and λ for $0 \leq \varepsilon \leq 1$ and $|\lambda| \leq K$, (1.7) implies

$$\begin{aligned} & \sup_{\|(1-\Delta)^{s/2} \nu\|_0 \leq 1} \int V(y) H_N(y) dy \\ & \leq c C \|h\|_{-1} + \sup_{\|(1-\Delta)^{s/2} \nu\|_0 \leq 1} \int V(y) (1 - \varphi(y)) H_N(y) dy. \end{aligned}$$

One also has $\sup_{y_1 \in \mathbb{R}} \|V(y_1, \cdot)\|_0 \leq C'\|v\|_0$, and $\|(1 - \Delta)^{S/2}v\|_0 \leq 1$ implies $\|v\|_0 \leq 1$. Thus an application of (1.5) (with $m = 1$) yields

$$(1.9) \quad \|(1 + y_1^2)^{-1}(1 - \rho(y_1))B_N(y)\|_{-S} \leq k\|h\|_{-1}$$

where k is independent of N , λ and ε .

Next we set

$$M' = (1/i)|((I^{-1})'(\eta))^*x - y|^{-2}(((I^{-1})'(\eta))^*x - y) \cdot \partial/\partial\eta.$$

Since $((I^{-1})'(\eta))^*x = ((\Lambda'(0\xi))^*)^{-1}0x = ((\partial\tau \circ 0^*/\partial\xi'_1(0\xi))^{-1}(0x)_1, \dots)$ where $\xi = I^{-1}(\eta)$, there is an $R_0 > 0$ such that $|((I^{-1})'(\eta))^*x - y| > 1$ for $(0x)_1 < -R_0$, $y_1 > -R'' - 1$, $\eta \in \tilde{C}$. Hence for $(0x)_1 < -R_0$ and r sufficiently large

$$(1.10) \quad \begin{aligned} & Q(D)u_N(x) \\ &= \left((2\pi)^{-n} \int \exp(i(x \cdot I^{-1}(\eta) - y \cdot \eta))(M')^{*r}(\bar{\psi}(\eta))d\eta, (1 - \rho(y_1))B_N \right) \\ &+ (2\pi)^{-n} \int \exp(ix \cdot I^{-1}(\eta))\bar{\psi}(\eta)d\eta \int \exp(-iy \cdot \eta)\rho(y_1)B_N(y)dy. \end{aligned}$$

Now (1.8), (1.9) and (1.10) imply for all m

$$\sup_{(0x)_1 \leq -R_0} \sum_{|\alpha| \leq m} |D^\alpha Q(D)u_N(x)| \leq K_m \|h\|_{-1}$$

and since $Q(D)u_N \rightarrow Q(D)u$ in $L^2(\mathbb{R}^n)$ as $N \rightarrow \infty$ and K_m is independent of N , we have

$$(1.11) \quad \sup_{(0x)_1 \leq -R_0} \sum_{|\alpha| \leq m} |D^\alpha Q(D)u(x)| < K_m \|h\|_{-1}.$$

To complete the proof we only need to observe

$$(x(t), \xi(t)) = (\hat{x} + (T - t)\partial\tau/\partial\xi(\hat{\xi}), \hat{\xi})$$

for $t < -T$. Since $(0\partial\tau/\partial\xi)_1 = (\partial\tau \circ 0^*)/\partial\xi'_1 > 0$, by taking T_0 sufficiently large we may assume $(0x(-T_0))_1 < -R_0$. Since $\|h\|_{-1} \leq c(\|g\|_{-1} + \|u\|_0^R)$, the conclusion of Lemma 1.2 now follows from (1.11).

The following argument reduces the proof of Theorem 1.1 to the preceding lemma and the theorem of Hörmander cited earlier:

Let $(x(t), \xi(t))$ be the bicharacteristic of Lemma 1.2. Choose $R' > R$ such that $\{x(t) \mid |t| \leq T_0\}$ is contained in $|x| < R' - 1$ and choose $f \in C_0^\infty(|x| < R')$ so that $f(x) = 1$ for $|x| \leq R' - 1$. Let $\varphi(x, \xi)$ be a symbol of order zero with support in $\{V \cap (x, \xi) \mid |x| \leq R' - 1\}$ such that $\varphi(x, \xi) = 1$ on a conic neighborhood of $\Gamma = \{(x(t), \xi(t)) \mid |t| \leq T_0\}$. Note that $\Gamma \subset \{(x, \xi) \mid \det A(x, \xi) = 0\}$. We

call an operator $\varphi_0(x, D)u = \varphi(x, D)fu$ a cutoff to Γ . Given two cutoffs φ_0 and ψ_0 we say $\psi_0 > \varphi_0$ if $\psi(x, \xi) = 1$ on the support of $\varphi(x, \xi)$.

Given cutoffs, φ_0 and ψ_0 , $\psi_0 > \varphi_0$, let $P(x, D)$ be the operator with symbol $\varphi(x, \xi)P(x, \xi)$ and set $R_0(x, D) = P(x, D)f$. Similarly $\tau(x, D)$ is the operator with symbol $\psi(x, \xi)\tau(x, \xi)$ and $\tau_0(x, D) = \tau(x, D)f$. By construction, given $Lu + (i\lambda \pm \varepsilon)u = g$,

$$(1.12) \quad \tau_0 R_0 u + M R_0 u + (i\lambda \pm \varepsilon) R_0 u = R_0 g + M(R_0 - I)u,$$

where $M = R_0 L - \tau_0 R_0$ is an operator of order zero.

Let $\bar{\varphi}_0$ and $\bar{\psi}_0$ be cutoffs such that $\bar{\varphi}_0 < \bar{\psi}_0 < \varphi_0$. Then

$$(1.13) \quad \begin{aligned} L(\bar{\varphi}_0(R_0 - I)u) &= -(i\lambda \pm \varepsilon)\bar{\varphi}_0(R_0 - I)u + [L, \bar{\varphi}_0(R_0 - I)]\bar{\psi}_0 u \\ &\quad + [L, \bar{\varphi}_0(R_0 - I)](I - \bar{\psi}_0)u + \bar{\varphi}_0(R_0 - I)g. \end{aligned}$$

Let Λ^s denote the operator with symbol $|\xi|^s \psi(x, \xi)$ and $\Lambda_0^s = \Lambda^s(x, D)f$. Since the symbol of $I - R_0$ is the orthogonal projection onto $N(x, \xi)^\perp$ on the support of $\bar{\varphi}_0$, $\|\Lambda_0^1 R_0(\bar{\varphi}_0(R_0 - I)u)\|_s \leq C_s(\|\bar{\psi}_0 u\|_s + \|u\|_0^{R'})$. However, since $A(x, \xi)$ is invertible on $N(x, \xi)^\perp$ for $(x, \xi) \in U$, for $(x, \xi) \in U$, $(\Lambda_0^s L)^*(\Lambda_0^s L) + (\Lambda_0^{s+1} R_0)^*(\Lambda_0^{s+1} R_0)$ is an elliptic operator on the support of $\varphi(x, \xi)f(x)$. Thus by Gårding's inequality

$$\begin{aligned} \|\bar{\varphi}_0(R_0 - I)u\|_{s+1}^2 &\leq C(\bar{\varphi}_0(R_0 - I)u, (\Lambda_0^s L)^*(\Lambda_0^s L)\bar{\varphi}_0(R_0 - I)u) \\ &\quad + C(\bar{\varphi}_0(R_0 - I)u, (\Lambda_0^{s+1} R_0)^*(\Lambda_0^{s+1} R_0)\bar{\varphi}_0(R_0 - I)u) + C(\|u\|_0^{R'})^2. \end{aligned}$$

Hence

$$(1.14) \quad \|\bar{\varphi}_0(I - R_0)u\|_{s+1} \leq c_s(\|L(\bar{\varphi}_0(I - R_0)u)\|_s + \|\Lambda_0^1 R_0(\bar{\varphi}_0(I - R_0)u)\|_s + \|u\|_0^{R'}).$$

Hence, combining (1.13) and (1.14) we have

$$(1.15) \quad \|\bar{\varphi}_0(R_0 - I)u\|_{s+1} \leq K_s(\|\bar{\psi}_0 u\|_s + \|u\|_0^{R'} + \|g\|_s)$$

where K_s is independent of ε and λ for $0 < \varepsilon \leq 1$, $|\lambda| < K$. Since $[\bar{\varphi}_0, M]$ is order -1 , if we rewrite (1.12) as $(\tau_0 + M + (i\lambda \pm \varepsilon)I)R_0 u = p$, (1.15) implies

$$(1.16) \quad \|\bar{\varphi}_0 p\|_{s+1} \leq c_s(\|\bar{\psi}_0 u\|_s + \|u\|_0^{R'} + \|g\|_{s+1}).$$

By Lemma 1.2

$$\|\varphi_\pm u\|_{s+1} \leq K_{s+1}(\|g\|_0 + \|u\|_0^{R'})$$

and the symbols $\varphi_\pm(x, \xi) = P(\xi)$ on N_\pm . Hence

$$(1.17) \quad \|\rho_\pm R_0 u\|_{s+1} \leq K_{s+1}(\|g\|_0 + \|u\|_0^{R'})$$

for suitably chosen ρ_\pm , pseudo-differential operators of order zero with $\rho_\pm(x, \xi) = 1$ on conic neighborhoods of $(x(\pm T), \xi(\pm T))$. Setting $s = 0$ in (1.16) and

(1.17), it now follows from Proposition 3.5.1 of [5] (see (1.1)) that there is a cutoff ρ_0 , $\rho_0 < \bar{\varphi}_0$, such that

$$(1.18) \quad \|\rho_0 P_0 u\|_1 \leq c(\|g\|_1 + \|u\|_0^{R'}).$$

In Proposition 3.5.1 Hörmander considers scalar operators. However, since the top order part of our operator, τ_0 , is scalar, the proof in [5] goes through unchanged.

Combining (1.15) and (1.18),

$$\|\rho_0 u\|_1 \leq c_1(\|g\|_1 + \|u\|_0^{R'}).$$

Choosing new cutoffs which we again call $\bar{\varphi}_0$ and $\bar{\psi}_0$ such that $\bar{\varphi}_0 < \bar{\psi}_0 < \rho_0$, (1.15) now implies

$$\|\bar{\varphi}_0(P_0 - I)\|_2 \leq K(\|g\|_1 + \|u\|_0^{R'}) \quad \text{and} \quad \|\bar{\varphi}_0 P\|_2 \leq K(\|g\|_2 + \|u\|_0^{R'}).$$

Using (1.17) with $s = 1$ and applying Hörmander's theorem again, we have a cutoff $\rho_1 < \bar{\varphi}_0$ such that $\|\rho_1 u\|_2 \leq c(\|g\|_2 + \|u\|_0^{R'})$. Repeating this argument, we conclude given s there is cutoff ρ_{s-1} such that

$$(1.19) \quad \|\rho_{s-1} u\|_s \leq C_s(\|g\|_s + \|u\|_0^{R'})$$

where C_s is independent of ε and λ for $0 < \varepsilon \leq 1$. In particular $\rho_{s-1}(x, \xi) = 1$ on a conic neighborhood of the original point (x_0, ξ_0) (see remarks preceding Lemma 1.2). Since (x_0, ξ_0) was an arbitrary point in S_R and S_R is compact, we can find a finite set of points $(x_0^i, \xi_0^i) \in S_R$, $i = 1, \dots, N$, such that $\sum_{i=1}^N (\rho_{s-1}^i(x, \xi))^2 \geq 1$ on a conic neighborhood of S_R . Since L is elliptic off any conic neighborhood of S , there is an $\alpha \in C_0^\infty(\mathbf{R}^n)$, $\alpha = 1$ for $|x| \leq R$, such that

$$(1.20) \quad \|\alpha u\|_s \leq C_s \left(\|(L + i\lambda + \varepsilon)u\|_{s-1} + \sum_{i=1}^N \|\rho_{s-1}^i(u)\|_s + \|u\|_0^{R'} \right).$$

Thus, combining (1.19) and (1.20), we have Theorem 1.1.

The hypothesis $\varepsilon > 0$ was only used in Lemma 1.2. Hence the proof of Theorem 1.1 also yields

Theorem 1.3. *Let $Lu + i\lambda u = g$, $g \in C_0^\infty(|x| < R)$, $u \in L^2(\mathbf{R})$, $u = 0$ for $|x| > R$. Then given s there is a C_s independent of λ and u such that*

$$\|\alpha u\|_s \leq C_s(\|g\|_s + \|u\|_0).$$

Theorem 1.3 enables us to prove the point spectrum of L is discrete (cf. [7, Theorem 3.1] and [10]).

Corollary 1.4. *The eigenvalues of L have no accumulation points and the nonzero eigenvalues have finite multiplicity.*

Proof. Since iL is selfadjoint, the eigenvalues of L are purely imaginary. Consider $f \in L^2(\mathbf{R}^n)$ with $(L + i\lambda)f = 0$, $\lambda \neq 0$. Let J_ϵ be the standard "mollification" operator. Then $J_\epsilon f \in L^2(\mathbf{R}^n) \cap C^\infty$ and

$$(L_0 + i\lambda)J_\epsilon f = (L_0 - L)J_\epsilon f + [L, J_\epsilon]f.$$

Since $[L, J_\epsilon]$ is a bounded operator on $L^2(\mathbf{R}^n)$ and $[L, J_\epsilon]v = 0$ for all v with support in $|x| > R + \epsilon$, $J_\epsilon f$ satisfies the hypotheses of Theorem 3.3 of [7]. This theorem is proved via the Radon transform and requires n odd. Hence $J_\epsilon f = 0$ for $|x| > R + \epsilon$. Since $J_\epsilon f \rightarrow f$ as $\epsilon \rightarrow 0$, we have $f = 0$ for $|x| > R$.

Suppose L has an infinite sequence of orthonormal eigenfactors f_n with eigenvalues $i\lambda_n$ and $\lambda_n \rightarrow \lambda$ where $\lambda_n \neq 0$ though λ may be. Then, by Theorem 1.1, $\|f_n\|_1 \leq C$ where C is independent of n . Since $\{f \mid \|f\|_1 \leq C, f = 0, |x| > R\}$ is precompact in L^2 , we have a contradiction which proves the corollary.

2. Construction of generalized eigenfunctions. In this section we will construct solutions to $Lu + i\lambda u = 0$, $\lambda \in \mathbf{R}$, by the limiting absorption method. These eigenfunctions will satisfy "radiation" and mild regularity conditions which will enable us to prove a strong decay result (Theorem 2.2) for two subspaces of initial data. In §3 we will show the sum of these subspaces is dense in $L^2(\mathbf{R}^n) \ominus H_0$.

Let $\tau_j(\omega)$ and $r_j(\omega)$, $j = 1, \dots, k$, be functions on the unit sphere S^{n-1} such that $\|r_j(\omega)\| = 1$, $\tau_1 \geq \tau_2 \geq \dots \geq \tau_k$, and for $A_0(\xi) = \sum_{j=1}^n A_j^0 \xi_j$ we have

$$A_0(\omega)r_j(\omega) = \tau_j(\omega)r_j(\omega) \quad \forall \omega \in S^{n-1}.$$

The functions $\tau_j(\omega)$ and $r_j(\omega)$ may be chosen measurable on S^{n-1} .

Since $\partial\tau/\partial\xi(x, \xi) \neq 0$ for (x, ξ) in a neighborhood of S and $\xi \cdot \partial\tau/\partial\xi(x, \xi) = 0$ on S , the set $S_0 = \{\omega \in S^{n-1} \mid \det A_0(\omega) = 0\}$ is a smooth submanifold of S^{n-1} of codimension 1 with a finite number of components. Hence S_0 is a closed subset of measure zero in S^{n-1} .

For $\xi \in \mathbf{R}^n - \{0\}$ let $\varphi_j(x, \xi) = e^{ix \cdot \xi} r_j(\xi/|\xi|)$. Then

$$(L - i\tau_j(\xi))\varphi_j(x, \xi) = (L - L_0)\varphi_j(x, \xi) \equiv g_j(x, \xi)$$

is C^∞ in x and has support in x contained in $|x| < R$. Let $v_{j,\epsilon}^+(x, \xi)$ and $v_{j,\epsilon}^-(x, \xi)$ be the square-integrable solutions to

$$(L - (i\tau_j(\xi) \pm \epsilon))v_{j,\epsilon}^\pm = -g_j,$$

and let $\|\cdot\|_s^R$ denote the Sobolev s -norm (s a positive integer) over the ball $|x| \leq R$.

By Theorem 1.1 for ξ in a compact set C in $\mathbf{R}^n - \{0\}$ and $0 < \epsilon \leq 1$

$$(2.1) \quad \|v_{j,\epsilon}^\pm(\cdot, \xi)\|_s^R \leq C_s(K_s + \|v_{j,\epsilon}^\pm(\cdot, \xi)\|_0^R)$$

where C_s is independent of ξ and ε and $K_s = \sup_{\xi \in C} \|g_j(\cdot, \xi)\|_s < \infty$. We set $M = \{\xi \in \mathbf{R}^n - \{0\} \mid \xi/|\xi| \notin S_0, \det(A_0(\xi) - \lambda I) \neq 0 \text{ for } i\lambda \text{ in point spectrum } L\}$.

By Corollary 1.4 the point spectrum of L has no accumulation points. Hence M is an open subset of \mathbf{R}^n and M^c has measure zero.

Lemma 2.1. *Let C be compact subset of M . Then*

$$\sup_{1 \geq \varepsilon > 0; \xi \in C} \|v_{j,\varepsilon}^+(\cdot, \xi)\|_0^{R'} < \infty \quad \text{and} \quad \sup_{1 \geq \varepsilon > 0; \xi \in C} \|v_{j,\varepsilon}^-(\cdot, \xi)\|_0^{R'} < \infty.$$

Proof. We will give the proof for $v_{j,\varepsilon}^+$. The proof for $v_{j,\varepsilon}^-$ is identical. Assume $\sup_{1 \geq \varepsilon > 0; \xi \in C} \|v_{j,\varepsilon}^+(\cdot, \xi)\|_0^{R'} = \infty$, and set $u_\varepsilon = v_{j,\varepsilon}^+ / \|v_{j,\varepsilon}^+\|_0^{R'}$. Then by Rellich's compactness criterion and a Cantor diagonal argument we conclude from (2.1): there are sequences $\varepsilon_i \rightarrow 0$, $\xi_i \rightarrow \xi_0$, and $u \in C^\infty(|x| < R)$ such that $\tau_j(\xi_i) \rightarrow \lambda \neq 0$, $\|v_{j,\varepsilon_i}^+(\cdot, \xi_i)\|_0^{R'} \nearrow \infty$, and

$$\|u_{\varepsilon_i}(x, \xi_i) - u\|_s^R \rightarrow 0 \quad \text{as } i \rightarrow \infty, \text{ for all } s.$$

Hence setting $h_i = (L_0 - (i\tau_j(\xi_i) + \varepsilon_i))u_{\varepsilon_i}$ we have

$$(2.2) \quad \|h_i - (L_0 - L)u\|_s \rightarrow 0$$

for all s , and $h_i = 0$ for $|x| > R$.

To study the convergence of u_{ε_i} for $|x| > R$ we use the Radon transform: for $f \in C_0^\infty(\mathbf{R}^n)$ define

$$Rf = \frac{1}{2} \left(\frac{-\partial}{\partial s} \right)^{(n-1)/2} \left(\int_{x \cdot \omega = s} f dA \right).$$

The mapping R extends to a unitary map of $L^2(\mathbf{R}^n)$ onto the even or odd functions in $L^2(\mathbf{R} \times S^{n-1})$ as $(n-1)/2$ is even or odd (cf. [7, Theorem 2.2]). Interpreting $\partial/\partial s$ in the sense of distributions, one computes

$$(2.3) \quad (A_0(\omega)\partial/\partial s - (i\tau_j(\xi_i) + \varepsilon_i))Ru_{\varepsilon_i} = Rh_i$$

where $Rh_i(s, \omega) = 0$, $|s| \geq R$. For $\omega \in S^{n-1} - S_0$ we choose $k(\omega)$ so that $\tau_1(\omega) \geq \dots \geq \tau_{k(\omega)}(\omega) > 0 > \tau_{k(\omega)+1}(\omega) \geq \dots \geq \tau_k(\omega)$. Then, setting $\lambda_i = \tau_j(\xi_i)$, (2.3) implies

$$(2.4) \quad \begin{aligned} Ru_{\varepsilon_i}(s, \omega) = & \sum_{l=1}^{k(\omega)} - \int_s^\infty \left(\exp\left(\frac{i\lambda_l + \varepsilon_i}{\tau_l(\omega)}\right)(s - \sigma) \right) \frac{\overline{\eta_l(\omega)} \cdot Rh_i(\sigma, \omega)}{\tau_l(\omega)} d\sigma \eta_l(\omega) \\ & + \sum_{l=k(\omega)+1}^k \int_{-\infty}^s \left(\exp\left(\frac{i\lambda_l + \varepsilon_i}{\tau_l(\omega)}\right)(s - \sigma) \right) \frac{\overline{\eta_l(\omega)} \cdot Rh_i(\sigma, \omega)}{\tau_l(\omega)} d\sigma \eta_l(\omega) \end{aligned}$$

for $\omega \in S^{n-1} - S_0$.

By (2.2) we have

$$\lim_{i \rightarrow \infty} \operatorname{Re} \int_{\mathbf{R}^n} \overline{u_{\varepsilon_i}} \cdot h_i = \lim_{i \rightarrow \infty} \operatorname{Re} \int_{\mathbf{R}^n} \overline{u_{\varepsilon_i}} \cdot (L_0 - L)u_{\varepsilon_i} = 0.$$

Hence since R is an isometry

$$(2.5) \quad 0 = \lim_{i \rightarrow \infty} \operatorname{Re} \int_{\mathbf{R} \times S^{n-1}} \overline{Ru_{\varepsilon_i}} \cdot Rh_i d\omega d\sigma.$$

Substituting (2.4) into (2.5), one obtains after brief manipulation

$$\begin{aligned} 0 = & \lim_{i \rightarrow \infty} \int_{S^{n-1}} \sum_{l=1}^k \frac{-1}{|\tau_l(\omega)|} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} d\sigma \\ & \cdot \exp(-\varepsilon_i |s - \sigma| / |\tau_l(\omega)|) (\exp(-i\lambda_i s / \tau_l(\omega)) Rh_i(s, \omega) \cdot \overline{\eta_l(\omega)}) \\ & \cdot (\exp(-i\lambda_i \sigma / \tau_l(\omega)) Rh_i(\sigma, \omega) \cdot \overline{\eta_l(\omega)}). \end{aligned}$$

Since $\exp((- \varepsilon_i / |\tau_l(\omega)|) |s - \sigma|)$ is a positive kernel and $\{Rh_i(\cdot, \omega)\}$ is converging in $L^2(ds)$ for all ω by (2.2), it follows by Fatou's lemma

$$0 = \sum_{l=1}^k \frac{-1}{|\tau_l(\omega)|} \left| \int_{-\infty}^{\infty} \exp(-i\lambda s / \tau_l(\omega)) Rh(s, \omega) \cdot \overline{\eta_l(\omega)} ds \right|^2$$

a.e. in ω , where $h = (L_0 - L)u$. Hence

$$(2.6) \quad 0 = \int_{-\infty}^{\infty} \exp(i\lambda s / \tau_l(\omega)) Rh(s, \omega) \cdot \eta_l(\omega) ds$$

a.e. in ω , $l = 1, \dots, k$.

Integrating by parts in (2.4) we have

$$\begin{aligned} (2.7) \quad Ru_{\varepsilon_i}(s, \omega) = & \sum_{l=1}^{k(\omega)} \left[\frac{\overline{\eta_l(\omega)} \cdot Rh_i(s, \omega) \eta_l(\omega)}{i\lambda_i + \varepsilon_i} \right. \\ & \left. + \int_s^{\infty} \left(\exp\left(\frac{i\lambda_i + \varepsilon_i}{\tau_l(\omega)}\right) (s - \sigma) \right) \frac{\overline{\eta_l(\omega)} \cdot \partial Rh_i / \partial \sigma}{i\lambda_i + \varepsilon_i} d\sigma \eta_l(\omega) \right] \\ & + \sum_{k(\omega)+1}^k \left[\frac{-\overline{\eta_l(\omega)} \cdot Rh_i(s, \omega) \eta_l(\omega)}{i\lambda_i + \varepsilon_i} \right. \\ & \left. + \int_{-\infty}^s \left(\exp\left(\frac{i\lambda_i + \varepsilon_i}{\tau_l(\omega)}\right) (s - \sigma) \right) \frac{\overline{\eta_l(\omega)} \cdot \partial Rh_i / \partial \sigma}{i\lambda_i + \varepsilon_i} d\sigma \eta_l(\omega) \right]. \end{aligned}$$

Since (2.2) implies Rh_i converges uniformly together with its derivatives to Rh and λ_i is bounded away from zero, it follows from (2.7) that $|Ru_{\varepsilon_i}(s, \omega)| < c \forall i$, $s \in \mathbf{R}$, $\omega \in S^{n-1} - S_0$. Moreover, $Ru_{\varepsilon_i}(s, \omega)$ converges pointwise on $\mathbf{R} \times \{S^{n-1} - S_0\}$ to

$$\begin{aligned}
(2.8) \quad v(s, \omega) &= \sum_{l=1}^{k(\omega)} - \int_s^\infty \exp\left(\frac{i\lambda}{\tau_l(\omega)}(s - \sigma)\right) \frac{\overline{\eta_l(\omega)} \cdot Rh(\sigma, \omega)}{\tau_l(\omega)} d\sigma \eta_l(\omega) \\
&+ \sum_{l=k(\omega)+1}^k \int_{-\infty}^s \exp\left(\frac{i\lambda}{\tau_l(\omega)}(s - \sigma)\right) \frac{\overline{\eta_l(\omega)} \cdot Rh(\sigma, \omega)}{\tau_l(\omega)} d\sigma \eta_l(\omega).
\end{aligned}$$

By (2.6), $v(s, \omega) \in L^2(\mathbf{R} \times S^{n-1})$. Making use of (a) R is an isometry, and (b) R maps $C_0^\infty(\mathbf{R}^n)$ to $C_0^\infty(\mathbf{R} \times S^{n-1})$, we conclude from the preceding that u_{e_i} converges in the sense of distributions to an extension \tilde{u} of u with $R\tilde{u} = v$, and in the sense of distributions $(L_0 - i\lambda)\tilde{u} = (L_0 - L)\tilde{u}$. Moreover, by the Lebesgue dominated convergence theorem Ru_{e_i} converges to $R\tilde{u}$ in $L^2(\{|s| < R'\} \times S^{n-1})$ and thus (see (2.12)) u_{e_i} converges to \tilde{u} in $L^2(|x| < R')$. Thus $\tilde{u} \neq 0$ and $i\lambda$ is an eigenvalue of L . This contradicts $C \subset \subset M$ and completes the proof.

Given a set $C \subset \subset M$, Lemma 2.1 and the estimate (2.1) imply for all s

$$(2.9) \quad \sup_{\xi \in C, 0 < \varepsilon \leq 1} \|v_{j,\varepsilon}^\pm(\cdot, \xi)\|_s^R < \infty.$$

Hence $\{v_{j,\varepsilon}^\pm(x, \xi) \mid 0 < \varepsilon \leq 1\}$ is a weakly compact set in $L^2(\{|x| < R\} \times C)$ for all $C \subset \subset M$. Choosing $C_m \nearrow M$, $C_m \subset \subset M$, and using a Cantor diagonal argument, we can choose $\varepsilon_i \rightarrow 0$ so that v_{j,ε_i}^\pm converges weakly to a function v_j^\pm on $\{|x| < R\} \times M$ in $L^2(\{|x| < R\} \times C_m)$ for all m . Since the weak closure of a set is contained within its closed convex hull v_j^\pm satisfies (2.9) with $s = 0$. [It is possible to choose v_j^\pm so that it satisfies (2.9) for all s , but we will not use this.]

From this point on we will consider only v_j^+ but our arguments will apply equally well to v_j^- . The formula (2.7) remains valid with $u_{e_i} = v_{j,\varepsilon_i}^+$, $\lambda_i = \tau_j(\xi)$, and $h_i = (L_0 - L)v_{j,\varepsilon_i}^+ - g_j(x, \xi)$ and we conclude

$$(2.10) \quad |Rv_{j,\varepsilon_i}^+(s, \omega, \xi)| < K_m, \quad s \in \mathbf{R}, \omega \in S^{n-1} - S_0$$

and $\xi \in C_m$. Thus, again using weak compactness and a Cantor diagonal argument, we conclude there is a subsequence of the Rv_{j,ε_i}^+ which converges weakly to a function Rv_j^+ on $\mathbf{R} \times S^{n-1} \times M$ in $L^2(\{|s| < m\} \times S^{n-1} \times C_m)$ for all m . We denote this subsequence again by Rv_{j,ε_i}^+ . Since the weak closure of a set is contained in its closed convex hull, we have $|Rv_j^+(s, \omega, \xi)| \leq K_m$ a.e. for $(s, \omega, \xi) \in \mathbf{R} \times S^{n-1} \times C_m$, and

$$\begin{aligned}
(2.11) \quad Rv_j^+(s, \omega, \xi) \cdot \overline{\eta_l(\omega)} &= 0, \text{ for } s > R, l = 1, \dots, k(\omega) \\
&\text{and for } s < -R, l = k(\omega) + 1, \dots, k, \text{ for } (s, \omega, \xi) \in \mathbf{R} \times S^{n-1} \times M.
\end{aligned}$$

The notation Rv_j^+ is justified as follows. For $\varphi \in C_0^\infty(\mathbf{R}^n)$ the inverse of R is given by

$$(2.12) \quad \varphi(x) = \frac{1}{2} \int_{S^{n-1}} \left(-\frac{\partial}{\partial s}\right)^{(n-1)/2} R\varphi(\omega \cdot x, \omega) d\omega$$

cf. [7, Lemma 2.1]. Hence the square-integrable functions on \mathbf{R}^n given by

$$u^n(x, \xi) = R^{-1}\{Rv_j^+ \upharpoonright |s| < n\},$$

$n = 1, \dots, \infty$, satisfy $u^n(x, \xi) = u^{n+1}(x, \xi)$ for $|x| < n$. Defining $u(x, \xi) = u^n(x, \xi)$ for $|x| \leq n$, we have $u(x, \xi) \in L_{\text{loc}}^2(\mathbf{R}^n \times M)$. Given $\psi(\xi) \in C_0^\infty(M)$ and $\varphi \in C_0^\infty(\mathbf{R}^n)$, we have

$$\begin{aligned} \int_{\mathbf{R}^n \times M} \psi(\xi) \overline{\varphi(x)} u(x, \xi) &= \int_{\mathbf{R} \times S^{n-1} \times M} \psi(\xi) \overline{R\varphi(s, \omega)} \cdot Rv_j^+(s, \omega, \xi) \\ (2.13) \quad &= \lim_{i \rightarrow \infty} \int_{\mathbf{R} \times S^{n-1} \times M} \psi(\xi) \overline{R\varphi(s, \omega)} \cdot Rv_{j, e_i}^+(s, \omega, \xi) \\ &= \lim_{i \rightarrow \infty} \int_{\mathbf{R}^n \times M} \psi(\xi) \varphi(x) v_{j, e_i}^+(x). \end{aligned}$$

Thus $v_j^+(x, \xi) = u(x, \xi)$ a.e. for $|x| < R$ and we define $v_j^+(x, \xi) = u(x, \xi)$ a.e. for $|x| > R$.

Let $V = \{f \in L^2(\mathbf{R}^n) \mid Rf \text{ has compact support}\}$, and set $r_j(\xi) = r_j(\xi/|\xi|)$, $\tau_j(\xi) = |\xi| r_j(\xi/|\xi|)$. We define an “outgoing representation” $\mathfrak{F}_+ f$ for $f \in V$ by

$$\begin{aligned} \mathfrak{F}_+ f &= \tilde{f}_+(\xi) = \hat{f}(\xi) + (2\pi)^{-n/2} \sum_{j=1}^k r_j(\xi) \int_{\mathbf{R} \times S^{n-1}} Rf(s, \omega) \cdot \overline{Rv_j^+(s, \omega, \xi)} \\ (2.14) \quad &= \sum_{j=1}^k r_j(\xi) \tilde{f}_{j, \delta}^+(\xi). \end{aligned}$$

The “incoming representation” \mathfrak{F}_- is defined analogously substituting v_j^- for v_j^+ . We will eventually show \mathfrak{F}_+ extends to an isometry from $L^2(\mathbf{R}^n) \ominus H_0$ onto $L^2(\mathbf{R}^n)$. The next theorem is the first step in this direction. Following Lax and Phillips we define “incoming” and “outgoing” subspaces $D^- = \{f \in L^2(\mathbf{R}^n) \mid Rf(s, \omega) \cdot \overline{r_j(\omega)} = 0 \text{ for } s < R, j = 1, \dots, k(\omega); Rf(s, \omega) \cdot \overline{r_j(\omega)} = 0 \text{ for } s > -R, j = k(\omega) + 1, \dots, k\}$, $D^+ = \{f \in L^2(\mathbf{R}^n) \mid Rf(s, \omega) \cdot \overline{r_j(\omega)} = 0 \text{ for } s > -R, j = 1, \dots, k(\omega), Rf(s, \omega) \cdot \overline{r_j(\omega)} = 0, s < R, j = k(\omega) + 1, \dots, k\}$ respectively.

Theorem 2.1. \mathfrak{F}_\pm extends to an isometry from $H_\mp = \{\overline{U(t)f} \mid f \in D_\mp, t \in \mathbf{R}\}$ onto $L^2(\mathbf{R}^n)$.

Proof. This proof follows [9, Theorem 2].

We give the proof for \mathfrak{F}_+ ; the proof for \mathfrak{F}_- is identical.

Since $R(U_0(t)f)(s, \omega) = \sum_{j=1}^k (Rf(s - \tau_j(\omega)t, \omega) \cdot \overline{r_j(\omega)}) r_j(\omega)$, $U_0(t)$ maps V onto V . $U(t)$ maps functions of compact support to functions of compact support. Letting $c_0 = \max_{|\omega|=1} \{|\tau_j(\omega)| \mid j = 1, \dots, k\}$, the standard domain of dependence argument shows $U(t)f = U_0(t)f$, if $f = 0$ for $|x| < R + c_0 t$. Thus $U(t)$ maps V onto V for all t .

Given $\varphi \in C_0^\infty(\mathbf{R}^n)$ set

$$\tilde{\varphi}_{j,\varepsilon}^+(t, \xi) = \int_{\mathbf{R}^n} U(t)\varphi \cdot \overline{[e^{ix\xi}f_j(\xi) + v_{j,\varepsilon}^+(x, \xi)]} dx.$$

Since $L(e^{ix\xi}f_j + v_{j,\varepsilon}^+) = i\tau_j(\xi)(e^{ix\xi}f_j + v_{j,\varepsilon}^+) + \varepsilon v_{j,\varepsilon}^+$ in the sense of distributions, one has

$$\frac{\partial \tilde{\varphi}_{j,\varepsilon}^+}{\partial t} = i\tau_j(\xi)\varphi_{j,\varepsilon}^+ - \varepsilon \int U(t)\varphi \cdot \overline{v_{j,\varepsilon}^+(x, \xi)} dx.$$

Thus

$$(2.15) \quad \tilde{\varphi}_{j,\varepsilon}^+(t, \xi) = \exp(i\tau_j(\xi)t)\tilde{\varphi}_{j,\varepsilon}^+(0, \xi) + \varepsilon \int_0^t \exp(i\tau_j(\xi)(t-t'))m(t')dt'$$

where $m(t) = -\int U(t)\varphi \cdot \overline{v_{j,\varepsilon}^+(x, \xi)} dx$. Since the Fourier transform is unitary, we can replace φ in (2.15) by an arbitrary $f \in V$. Then we have for $\psi \in C_0^\infty(M)$

$$(2.16) \quad \begin{aligned} & \int \psi(\xi)\tilde{f}_{j,\varepsilon_i}^+(t, \xi) d\xi \\ &= \int \psi(\xi)\exp(i\tau_j(\xi)t)\tilde{f}_{j,\varepsilon_i}^+(0, \xi) d\xi \\ &+ \varepsilon_i \int_0^t dt' \int_{\mathbf{R} \times S^{n-1} \times M} \exp(i\tau_j(\xi)(t-t'))\psi(\xi)RU(t)f \cdot \overline{Rv_{j,\varepsilon_i}^+} d\xi. \end{aligned}$$

Note that the proof showing $U(t)$ maps V to V implies given $f \in V$ and $[-T, T]$ there is a ρ such that $R(U(t)f)(s, \omega) = 0$ for $|s| > \rho$ and $t \in [-T, T]$. Thus, using the estimate (2.10), we see that the last integral in (2.16) tends to zero as $i \rightarrow \infty$. Since Rv_{j,ε_i}^+ converges weakly to Rv_j^+ in $L^2(\{|s| < m\} \times S^{n-1} \times C_m)$ for all m , we have for all $\psi \in C_0^\infty(M)$

$$\int_M \psi(\xi)\tilde{f}_{j,0}^+(t, \xi) d\xi = \int_M \psi(\xi)\exp(i\tau_j(\xi)t)\tilde{f}_{j,0}^+(0, \xi) d\xi.$$

Hence for $f \in V$

$$(2.17) \quad (U(t)f)_+(t, \xi) = \sum_{j=1}^k \exp(i\tau_j(\xi)t)(\tilde{f}_+(\xi) \cdot \overline{\tilde{f}_+(\xi)})f_j(\xi)$$

a.e. in ξ .

Moreover, given $f \in D_- \cap V$, (2.11) implies $\tilde{f}_+(\xi) = \hat{f}(\xi)$, and by construction $D_- \cap V$ is dense in D_- . Thus by (2.17) for $f \in D_- \cap V$, $(U_0(t))f^\wedge(t, \xi) = (U(t))f_+^\wedge(t, \xi)$, and \mathcal{F}_+ extends to an isometry on H_- . Since the Fourier transform is a unitary map of L^2 onto L^2 , it suffices to show $\{U_0(t)D_- \mid t \in \mathbf{R}\}$ is dense in L^2 to complete the proof. However,

$$(RU_0(t)f)(s, \omega) = \sum_{j=1}^k (Rf(s + \tau_j(\omega)t)) \cdot \overline{(\tilde{f}_j(\omega))}f_j(\omega),$$

and, by Lemma 2.2 of [7], R^{-1} maps the even functions in $L^2(\mathbf{R}, L^2(S^{n-1}))$ onto $L^2(\mathbf{R}^n)$. Since $\prod_{j=1}^k \tau_j(\omega) = \det A_0(\omega) \neq 0$ on S_0^c and S_0 is a closed set of spherical measure zero, the desired conclusion follows easily.

The next theorem is the main ingredient in the proof of the existence of the scattering matrix.

Theorem 2.2. *Let P_{\pm} be the projection on D_{\pm}^{\perp} . Then, for f in H_- or H_+ , $\|P_{\pm} U(t)f\|_0 \rightarrow 0$ as $t \rightarrow \pm\infty$.*

Proof. As before we only consider $f \in H_-$. Since $C_0^{\infty}(M)$ is dense in $L^2(\mathbf{R}^n)$, Theorem 2.1 implies $W \equiv \mathcal{F}_+^*(C_0^{\infty}(M))$ is dense in H_- . We begin by showing $\|P_+ U(t)f\|_0 \rightarrow 0$ as $t \rightarrow \pm\infty$.

For $f \in V \cap H_-$ and $q \in C_0^{\infty}(M)$, (2.17) implies

$$\begin{aligned} (U(t)\mathcal{F}_+^* q, f) &= (q, \mathcal{F}_+ U(-t)f) = \sum_{j=1}^k \int (q(\xi) \cdot \overline{r_j(\xi)} \exp(i\tau_j(\xi)t)) r_j(\xi) \cdot \overline{\mathcal{F}_+ f(\xi)} d\xi \\ &= \sum_{j=1}^k \int (q(\xi) \cdot \overline{r_j(\xi)} \exp(i\tau_j(\xi)t)) \overline{\hat{f}(\xi)} \cdot r_j(\xi) d\xi \\ &\quad + \sum_{j=1}^k (2\pi)^{-n/2} \int (q(\xi) \cdot \overline{r_j(\xi)} \exp(i\tau_j(\xi)t)) \\ &\quad \cdot \left(\int_{\mathbf{R} \times S^{n-1}} \overline{Rf} \cdot Rv_j^+(s, \omega, \xi) d\sigma d\omega \right) d\xi \quad \text{by (2.14)} \\ &= (2\pi)^{-n/2} \int_{\mathbf{R}^n} \left(\sum_{j=1}^k \int_{\mathbf{R}^n} (q(\xi) \cdot \overline{r_j(\xi)}) \exp(i(\xi \cdot x + \tau_j(\xi)t)) d\xi \right) \cdot \overline{f(x)} dx \\ &\quad + (2\pi)^{-n/2} \int_{\mathbf{R} \times S^{n-1}} \left(\int_{\mathbf{R}^n} \sum_{j=1}^k q(\xi) \cdot \overline{r_j(\xi)} \exp(i\tau_j(\xi)t) Rv_j^+(s, \omega, \xi) d\xi \right) \\ &\quad \cdot \overline{Rf}(s, \omega) ds d\omega \end{aligned}$$

by Fubini's theorem, since $Rv_j^+ \in L_{\text{loc}}^2(\mathbf{R} \times S^{n-1} \times M)$. Hence, since $\{Rf | f \in V \cap H_-\}$ is dense in RH_- ,

$$\begin{aligned} (2\pi)^{n/2} RU(t)\mathcal{F}_+^* q &= R \left(\sum_{j=1}^k \int_{\mathbf{R}^n} (q(\xi) \cdot \overline{r_j(\xi)}) \exp(i(\xi \cdot x + \tau_j(\xi)t)) r_j(\xi) d\xi \right) \\ &\quad + \left(\sum_{j=1}^k \int_{\mathbf{R}^n} (q(\xi) \cdot \overline{r_j(\xi)} \exp(i\tau_j(\xi)t)) Rv_j^+(s, \omega, \xi) d\xi \right) \end{aligned}$$

and, letting $b(x) = \int q(\xi) e^{i\xi \cdot x} d\xi$, we have

$$\begin{aligned} (2\pi)^{n/2} RU(t)\mathcal{F}_+^* q &= \left[\left(\sum_{j=1}^k Rb(s + \tau_j(\omega)t, \omega) \cdot \overline{r_j(\omega)} r_j(\omega) \right) \right] \\ (2.18) \quad &\quad + \left[\sum_{j=1}^k \int_{\mathbf{R}^n} (q(\xi) \cdot \overline{r_j(\xi)} \exp(i\tau_j(\xi)t)) \overline{Rv_j^+(s, \omega, \xi)} d\xi \right]. \end{aligned}$$

Since $Rb(s, \omega)$ is an element of $L^2(\mathbf{R} \times S^{n-1})$, the L^2 -norm of the restriction of the first term on the right of (2.18) to $\{(s, \omega) \mid |s| < R\}$ tends to zero as $t \rightarrow \pm\infty$. To show the same is true for the second term it suffices to show for $j = 1, \dots, k$

$$(2.19) \quad 0 = \lim_{t \rightarrow \pm\infty} \left\| \int_{\mathbf{R}^n} (q(\xi) \cdot \overline{r_j(\xi)} \exp(i\tau_j(\xi)t)) Rv_j^+(s, \omega, \xi) d\xi \right\|_{L^2(\{|s| < R\} \times S^{n-1})}.$$

This will be a consequence of our estimates on Rv_j^+ , and a number of applications of the Riemann-Lebesgue lemma and the Lebesgue dominated convergence theorem. Introducing polar coordinates in ξ -space, the function of (2.19) becomes

$$(2.20) \quad H(s, \omega, t) = \int_{S^{n-1}} d\omega' \int_{\mathbf{R}} (q(\sigma\omega') \cdot \overline{r_j(\omega')} \exp(i\sigma\tau_j(\omega')t)) Rv_j^+(s, \omega, \sigma\omega') \sigma^{n-1} d\sigma.$$

Since $q(\xi) \in C_0^\infty$ and $|\tau_j(\xi)/|\xi|| \geq \delta > 0$ on support q , the Riemann-Lebesgue lemma implies for any sequence $t_i \rightarrow \pm\infty$ the inner integral of (2.20) tends to zero a.e. in ω', s, ω . Hence by Lebesgue dominated convergence $H(s, \omega, t_i) \rightarrow 0$ a.e. as $i \rightarrow \infty$. However, since $|Rv_j^+(s, \omega, \xi)| \leq C$ for $(s, \omega) \in \mathbf{R} \times S^{n-1}$ and ξ in support q , the restriction of $H(s, \omega, t_i)$ to $|s| < R$ is uniformly bounded by a constant multiple of the characteristic function of $\{s \mid |s| < R\}$ and another application of L.D.C. gives (2.19). Thus, for $f \in W$, $\|P_\pm U(t)f\|_0 \rightarrow 0$ as $t \rightarrow \pm\infty$.

From this point the proof may be identified with a subset of the proof of [8, Lemma 3.8]. For $f \in W$ we consider $R(U(t)f)$. Since $U(t)f$ is a distribution solution to $(\partial/\partial t - L_0)U(t)f = (L - L_0)U(t)f$, and $(L - L_0)U(t)f$ is supported in $|x| < R$, we conclude $R(U(t)f)$ is a distribution solution to $(\partial/\partial t - A_0(\omega)\partial/\partial s)R(U(t)f) = 0$ in $|s| > R$. Hence

$$\sum_{k=1}^{k(\omega)} ((RU(t)f)(s, \omega) \cdot \overline{r_k(\omega)})r_k(\omega) = \sum_{l=1}^{k(\omega)} (Rf(s + \tau_l(\omega)t, \omega) \cdot \overline{r_l(\omega)})r_l(\omega)$$

for $t < 0, s < -R$, and for $t > 0, s > R$. Likewise for $t > 0, s < -R$, and $t < 0, s > R$

$$\sum_{l=k(\omega)+1}^k ((RU(t)f)(s, \omega) \cdot \overline{r_l(\omega)})r_l(\omega) = \sum_{l=k(\omega)+1}^k ((Rf)(s + \tau_l(\omega)t, \omega) \cdot \overline{r_l(\omega)})r_l(\omega).$$

Since f and hence Rf are square-integrable, $\prod_{j=1}^k \tau_j(\omega) = \det A_0(\omega) \neq 0$ on S_0^c , and S_0 is a closed set of spherical measure zero, it follows $\|P_{D_-} U(t)f\|_0 \rightarrow 0$ as $t \rightarrow \infty$ and $\|P_{D_+} U(t)f\|_0 \rightarrow 0$ as $t \rightarrow -\infty$. Here P_{D_\pm} is the projection on D_\pm . Combining this with $\|P_\pm U(t)f\|_0 \rightarrow 0$ as $t \rightarrow \pm\infty$, completes the proof of Theorem 2.2.

3. Unitarity of the scattering matrix. We begin by constructing a pseudo-differential operator-valued function of t , $Q(t)$. The use of $Q(t)$ in this section

could probably be avoided by another application of Hörmander's Proposition 3.5.1 but $Q(t)$ seems essential for the proof of Theorem 4.1. Our construction was suggested by that of [3, §2.4].

The symbol of $Q(t)$ will be given by $a(x, \xi, t)P(x, \xi) + Q_{-1}(x, \xi, t)/i \equiv Q_0 + Q_{-1}/i$, where Q_j is homogeneous degree j and $P(x, \xi)$ is projection operator of §1.

We want $[\partial/\partial t - L, Q(t)]$ to be an operator of order -1 . Thus we require

$$(3.1) \quad \begin{aligned} & A(x, \xi)Q_{-1}(x, \xi, t) - Q_{-1}(x, \xi, t)A(x, \xi) \\ &= \frac{\partial a}{\partial t}P + \sum_{i=1}^n \left(\frac{\partial Q_0}{\partial \xi_i} \frac{\partial A}{\partial x_i} - \frac{\partial A}{\partial \xi_i} \frac{\partial Q_0}{\partial x_i} \right) + Q_0 B - BQ_0 \equiv H. \end{aligned}$$

Let $P' = 1 - P$. As a first step in the solution of (3.1) for Q_{-1} , we show

$$'Q_{-1} \equiv \sum_{i=1}^n \frac{\partial P'}{\partial \xi_i} Q_0 \frac{\partial P'}{\partial x_i}$$

is a smooth solution to

$$(3.2) \quad A'Q_{-1} - 'Q_{-1}A = P'HP'.$$

We have

$$\begin{aligned} P'HP' &= \sum_{i=1}^n P' \frac{\partial Q_0}{\partial \xi_i} \frac{\partial A}{\partial x_i} P' - P' \frac{\partial A}{\partial \xi_i} \frac{\partial Q_0}{\partial x_i} P' \\ &= \sum_{i=1}^n \left[-\frac{\partial P'}{\partial \xi_i} Q_0 \frac{\partial A}{\partial x_i} P' + P' \frac{\partial A}{\partial \xi_i} Q_0 \frac{\partial P'}{\partial x_i} \right], \end{aligned}$$

since $P'P = 0$.

$$P'HP' = \sum_{i=1}^n -\frac{\partial P'}{\partial \xi_i} \left(-\frac{\partial Q_0}{\partial x_i} A + \frac{\partial \tau Q_0}{\partial x_i} \right) P' + P' \left(-A \frac{\partial Q_0}{\partial \xi_i} + \frac{\partial \tau Q_0}{\partial \xi_i} \right) \frac{\partial P'}{\partial x_i},$$

since $Q_0 A - \tau Q_0 = A Q_0 - \tau Q_0 = 0$.

$$\begin{aligned} P'HP' &= \sum_{i=1}^n \frac{\partial P'}{\partial \xi_i} \frac{\partial Q_0}{\partial x_i} P' A - A P' \frac{\partial Q_0}{\partial \xi_i} \frac{\partial P'}{\partial x_i} \\ &\quad + \sum_{i=1}^n \tau \left(-\frac{\partial P'}{\partial \xi_i} \frac{\partial Q_0}{\partial x_i} P' + P' \frac{\partial Q_0}{\partial \xi_i} \frac{\partial P'}{\partial x_i} \right) \\ &= \sum_{i=1}^n -\frac{\partial P'}{\partial \xi_i} Q_0 \frac{\partial P'}{\partial x_i} A + A \frac{\partial P'}{\partial \xi_i} Q_0 \frac{\partial P'}{\partial x_i}. \end{aligned}$$

Hence $'Q_{-1} = \sum_{i=1}^n \partial P'/\partial \xi_i Q_0 \partial P'/\partial x_i$ is a solution to (3.2) as claimed. If $PHP = 0$, there is a solution $'Q_{-1}$ to $(A - \tau I)'Q_{-1} = HP$ satisfying $'Q_{-1}P' = 0$. Hence $A'Q_{-1} - 'Q_{-1}A = HP$. Likewise, if $PH^*P = 0$, there is a solution Q'_{-1} to $Q'_{-1}(\tau I - A) = PH$ satisfying $P'Q' = 0$. Hence $AQ'_{-1} - Q'_{-1}A = PH$. A com-

putation similar to that following (3.2) shows both $PHP = 0$ and $PH^*P = 0$ reduce to

$$(3.3) \quad 0 = \frac{\partial a}{\partial t} + \sum_{i=1}^n \frac{\partial a}{\partial \xi_i} \frac{\partial \tau}{\partial x_i} - \frac{\partial a}{\partial x_i} \frac{\partial \tau}{\partial \xi_i}.$$

Hence, if (3.3) holds we may solve (3.1) with $Q_{-1} = 'Q'_{-1} + Q'_{-1} + 'Q_{-1}$ and Q_{-1} will be a symbol of order -1 depending smoothly on t . Given a bicharacteristic $(x(t), \xi(t))$ satisfying $\dot{x} = -\partial\tau/\partial\xi$ and $\dot{\xi} = \partial\tau/\partial x$, (3.3) merely states

$$(3.4) \quad 0 = da(x(t), \xi(t), t)/dt.$$

Thus we choose $a(x, \xi, t)$ constant on the curves $(x(t), \xi(t), t)$. If we choose $a(x, \xi, 0) = 1$ on S_R with $a(x, \xi, 0) = 0$ outside a sufficiently small neighborhood of S_R and determine a for $t > 0$ by (3.4), it follows by hypothesis (ii) that there is a T_0 such that $a(x, \xi, T_0) = 0$ for $|x| < R + 1$. We extend $a(x, \xi, t)$ to be homogeneous degree 0 in ξ for $|\xi| > 1$ and smooth in ξ on \mathbf{R}^n . The symbol $Q_{-1}(t, x, \xi)$ may be chosen to have support contained in the support of a .

We form $Q(t)$ with symbol $aP + Q_{-1}/i$ constructed in the preceding paragraph and given R' , have $\|[\partial/\partial t - L, Q(t)]u\|_{R'} \leq C_{R'} \|u\|_0$ for all $u \in C_0^\infty(|x| < R')$, where $C_{R'}$ is independent of t for $t \in [0, T_0]$. Moreover, multiplying $Q(t)$ on the right by $\psi(x, t)$, $\psi(x, t) = 0$ outside small neighborhood of support $a(x, \cdot, t)$, $\psi(x, t) = 1$ on support $a(x, \cdot, t)$, we may assume $Q(T_0)u = 0$ for all $u \in C_0^\infty(|x| < R)$, and $Q(t)u$ has compact support for all t .

We are now ready to complete the proof of the unitarity of the scattering matrix and energy decay. The following lemma is patterned on Theorem 1 of [9].

Lemma 3.1. $H_+ + H_-$ is dense in $L^2(\mathbf{R}^n) \ominus H_0$.

Proof. Suppose we have $q \in L^2(\mathbf{R}^n)$ such that $q \perp (H_0 + H_+ + H_-)$. Then, by the definition of H_+ and H_- , $U(t)q \perp D_+ \forall t$ and $U(t)q \perp D_-$ all t . Hence $R(U(t)q)(s, \omega) = 0$, $|s| > R, \forall t$. By domain of dependence $U_0(t - t_0)U(t_0)q = U(t)q$ for $|x| > R + 2c\delta$ and $|t - t_0| < \delta$, where c is the maximum sound speed of L , $c = \sup_{x \in \mathbf{R}^n} \{v \cdot A(x, \xi)v \mid |\xi| = 1, |v| = 1\}$. Let

$$h(r) \in (C_0^\infty(R + 2c\delta, \infty))^k$$

and consider

$$m(s, \omega) = \int_{\omega \cdot x = s} Y_m\left(\frac{x}{|x|}\right) h(|x|) dA$$

where Y_m is a spherical harmonic of degree m . Rotating coordinates so that $\omega \cdot x = y_1$

$$m(s, \omega) = \sum_{|a|=m} a_a(\omega) \int_{y_1=s} \frac{y^\alpha}{|y|^m} h(|y|) dy_2 dy_n.$$

Introducing polar coordinates for y_2, \dots, y_n and setting $\alpha = (\alpha_1, \alpha')$

$$m(s, \omega) = \sum_{|\alpha|=m} a_\alpha(\omega) s^{\alpha_1} \int_{S^{n-2}} d\omega' f_\alpha(\omega') \int_0^\infty (r^2 + s^2)^{-m/2} h((r^2 + s^2)^{1/2}) r^{n+|\alpha'|-2} dr.$$

Now if $|s| < R + c\delta$ the r -integration does not include $r = 0$ and we can introduce $u = (r^2 + s^2)^{1/2}$ and get

$$(3.5) \quad m(s, \omega) = \sum_{|\alpha|=m} b_\alpha(\omega) s^{\alpha_1} \int_{R+2c\delta}^\infty u^{(1-m)} h(u) (u^2 - s^2)^{(n+|\alpha'|-3)/2} du.$$

Now let $n(s, \omega, t_0)$ be the Radon transform of $U(t_0)q$. Then the Radon transform of $U_0(t - t_0)U(t_0)q$ is

$$\sum_{j=1}^n n_j(s + \tau_j(\omega)(t - t_0), \omega, t_0) f_j(\omega)$$

(here $n_j = n(s, \omega, t_0) \cdot \overline{f_j(\omega)}$). Letting

$$l_j(s, \omega, t_0) = \left(\frac{\partial}{\partial s} \right)^{(n-1/2)} n_j(s, \omega, t) \quad \text{and} \quad k_j(s, \omega) = \left(\frac{\partial}{\partial s} \right)^{(n-1/2)} m_j(s, \omega),$$

we have

$$\begin{aligned} C(t) &= \int h(x) Y_m \left(\frac{x}{|x|} \right) \overline{(U(t)q)(x)} = \int R(h Y_m) \overline{R(U(t)q)} \\ &= \sum_{j=1}^k \int_{S^{n-1}} d\omega \int_{|s - \tau_j(\omega)(t - t_0)| \leq R} k_j(s, \omega) \overline{l_j(s + \tau_j(\omega)(t - t_0), \omega, t_0)} ds \\ &= \sum_{j=1}^k \int_{S^{n-1}} d\omega \int_{|\nu| \leq R} k_j(\nu - \tau_j(\omega)(t - t_0), \omega) \overline{l_j(\nu, \omega, t_0)} d\nu \end{aligned}$$

and by (3.5) for $|\operatorname{Re}(t - t_0)| < \delta$ we have $C(t)$ analytic. Since t_0 was arbitrary, $C(t)$ is entire. Moreover, since the norm of $l(s, \omega, t_0)$ in $L^2(\mathbf{R} \times S^{n-1})$ is constant and

$$\sup_{|\nu| \leq R; |\omega|=1} |k_j(\nu - \tau_j(\omega)(t - t_0))| \leq C|t - t_0|^{(n-5)/2+m} \leq C(|\operatorname{Im} t| + \delta)^{(n-5)/2+m},$$

we conclude $C(t)$ is a polynomial. Since $C(t)$ is bounded on the real axis, $C(t)$ is constant. Thus $U(t)q(x)$ is constant for $|x| > R$.

Let $h = U(t_1)q - U(t_2)q$. Then $U(t)h = 0$ for $|x| > R$. If h is not zero, neither is $\varphi = n \int_0^{1/n} U(t)h dt$ for n sufficiently large. Moreover, $\varphi \in \mathfrak{D}(L)$ and $U(t)\varphi = 0$ for $|x| > R$.

Now let $Q(t)$ be the operator-valued function constructed at the beginning of this section. We denote the operator $-\partial/\partial t - L, Q(t)$ by $K(t)$. Then we have

$\partial Q(t - t_0)U(t)\varphi / \partial t - L(Q(t - t_0)U(t)\varphi) = K(t - t_0)U(t)\varphi$ and by Duhamel's formula

$$(3.6) \quad Q(t - t_0)U(t)\varphi = U(t - t_0)Q(0)U(t_0)\varphi + \int_{t_0}^t U(t - s)K(s - t_0)U(s)\varphi ds.$$

For all real s , $U(t)$ satisfies the estimate

$$(3.7) \quad \|U(t)f\|_s \leq C_s e^{\omega_s |t|} \|f\|_s.$$

To see this, assume first $s \leq 0$, and let Λ_s denote the operator on $L^2(\mathbf{R}^n)$ which multiplies the Fourier transform by $(1 + |\xi|^2)^{s/2}$. Then for $f \in \mathcal{D}(L)$

$$(3.8) \quad \frac{\partial \Lambda_s U(t)f}{\partial t} = L\Lambda_s U(t)f + [\Lambda_s, L]\Lambda_s^{-1} \Lambda_s U(t)f.$$

Since $[\Lambda_s, L]\Lambda_s^{-1} = [\Lambda_s, L - L_0]\Lambda_s^{-1}$ is an operator of order zero, and, for $|x| > R$, $([\Lambda_s, L]\Lambda_s^{-1}f)(x) = (\Lambda_s(L - L_0)\Lambda_s^{-1}f)(x)$, $[\Lambda_s, L]\Lambda_s^{-1}$ is a bounded operator on $L^2(\mathbf{R}^n)$. Thus $L + [\Lambda_s, L]\Lambda_s^{-1}$ is a bounded perturbation of the group generator L , and therefore generates an exponentially bounded group $\tilde{U}(t)$. Since $\Lambda_s U(t)f$ is continuously strongly differentiable, (3.8) implies $\Lambda_s U(t)f = \tilde{U}(t)\Lambda_s f$ and we have

$$(3.9) \quad \|\Lambda_s U(t)f\|_0 \leq C_s e^{\omega_s |t|} \|\Lambda_s f\|_0.$$

Since $C_0^\infty(\mathbf{R}^n) \subset \mathcal{D}(L)$ is dense in H_s , (3.7) follows immediately from (3.9).

For $s > 0$ we argue by duality:

$$\begin{aligned} \|U(t)f\|_s &= \sup_{g \in C_0^\infty; \|g\|_{-s} \leq 1} \left| \int \bar{g} U(t)f \right| = \sup \left| \int \overline{U(-t)g} f \right| \\ &\leq C_{-s} e^{\omega_{-s}|t|} \sup_{g \in C_0^\infty; \|g\|_{-s} \leq 1} \left| \int \bar{g} f \right| = C_{-s} e^{\omega_s |t|} \|f\|_s. \end{aligned}$$

Thus we have (3.7) for all s .

Since $\|K(s - t_0)U(s)\varphi\|_1 \leq C\|\varphi\|_0$, for $t_0 \leq s \leq T_0 + t_0$, we conclude from (3.6) and (3.7) with $s = 1$

$$\|Q(0)U(t_0)\varphi\|_1 \leq K_1 \|U(T_0)Q(0)U(t_0)\varphi\|_1 \leq K_2 (\|Q(T_0)U(t_0 + T_0)\varphi\|_1 + \|\varphi\|_0).$$

$U(t_0 + T_0)\varphi = 0$ for $|x| > R$ and by construction $Q(T_0)U(t_0 + T_0)\varphi = 0$. Hence $\|Q(0)U(t)\varphi\|_1 \leq K_2 \|\varphi\|_0 = C_0$. Since $\varphi \in \mathcal{D}(L)$, $\|LU(t)\varphi\|_0 = \|U(t)L\varphi\|_0 \leq \|L\varphi\|_0 = C_1$. We may choose $\alpha \in C_0^\infty$, $\alpha = 1$ for $|x| < R$, such that (compare (1.14)) $\|\alpha\psi\|_1 \leq C(\|L\psi\|_0 + \|Q(0)\psi\|_1)$. Thus we have

$$\|U(t)\varphi\|_1 = \|\alpha U(t)\varphi\|_1 \leq C(C_1 + C_0) \quad \forall t,$$

and $\{U(t)\varphi \mid t \in \mathbf{R}\}$ is a precompact set in $L^2(\mathbf{R}^n)$. Since $\varphi \perp H_0$, Lemma V.2.3

of [6] implies there are $T_i \rightarrow \infty$ such that $U(T_i)\varphi$ converges weakly to zero. By compactness $\|U(T_{i_k})\varphi\|_0 \rightarrow 0$ for a subsequence T_{i_k} and $\varphi = 0$. Thus $U(t)q = q \forall t$ and, since $q \perp H_0$, $q = 0$.

We define wave operators W_+ and W_- by

$$W_{\pm}f = \lim_{t \rightarrow \pm\infty} U(-t)U_0(t)f.$$

These limits exist for $f \in \{U_0(t)D_{\pm} \mid t \in \mathbf{R}\}$. Since the argument following (2.17) shows $\{\overline{U_0(t)f} \mid f \in D_+\} = \{\overline{U_0(t)f} \mid f \in D_-\} = L^2(\mathbf{R}^n)$, W_+ and W_- are isometries from $L^2(\mathbf{R}^n)$ into $L^2(\mathbf{R}^n)$.

Theorem 3.2. $\text{Range } W_+ = \text{Range } W_- = L^2(\mathbf{R}^n) \ominus H_0$.

Proof. The proofs for W_+ and W_- are the same; we give the proof for W_+ .

Let $h = q_1 + q_2$, $q_1 \in H_-$, $q_2 \in H_+$. Then, by Theorem 2.2, $\|P_+ U(t)h\|_0 < \varepsilon$ for $t \geq T$. Letting $(I - P_+)U(T)h = f$ we have for $t > 0$

$$\|U_0(t)f - U(t+T)h\|_0 < \varepsilon,$$

$$\|U(-t-T)U_0(t+T)[U_0(-T)f] - h\|_0 < \varepsilon,$$

$$\|W_+ U_0(-T)f - h\|_0 < \varepsilon.$$

Since ε is arbitrary and the range of W_+ is closed, we have $h \in \text{Range } W_+$. By Lemma 3.1, $\text{Range } W_+ \subset L^2(\mathbf{R}^n) \ominus H_0$. $f \in \text{Range } W_+ \cap H_0 \Rightarrow \|e^{i\lambda t}f - U_0(t)h\|_0 \rightarrow 0$ for some $h \in L^2(\mathbf{R}^n)$. Since $\|U_0(t)h\|_0^r \rightarrow 0$ as $t \rightarrow \infty$ for all r (again see argument following (2.17)), we conclude $f = 0$.

Corollary 3.3. $H^+ = H^- = L^2(\mathbf{R}^n) \ominus H_0$. Hence \mathcal{F}_+ and \mathcal{F}_- extend to unitary maps of $L^2(\mathbf{R}^n) \ominus H_0$ to $L^2(\mathbf{R}^n)$.

Corollary 3.4. The scattering matrix $S = W_-^* W_+$ is unitary.

Corollary 3.5. For $f \in L^2(\mathbf{R}^n) - H_0$ and $r < \infty$,

$$\|U(t)f\|_0^r \rightarrow 0, \quad t \rightarrow \pm\infty.$$

Proof. By Theorem 3.2 $\exists h_{\pm} \in L^2(\mathbf{R})$ such that $\|U(t)f - U_0(t)h_{\pm}\|_0 \rightarrow 0$ as $t \rightarrow \pm\infty$ and $\|U_0(t)h_{\pm}\|_0^r \rightarrow 0$ as $t \rightarrow \pm\infty$.

4. Concentration near bicharacteristics and an example of nondecay. In this section we assume the existence of a smooth real-valued function $\tau(x, \xi)$ and a smooth projection-valued function $P(x, \xi)$, defined on a conic open set V in (x, ξ) -space, such that $A(x, \xi)P(x, \xi) = \tau(x, \xi)P(x, \xi)$. Given a fixed $\lambda_0 \in \mathbf{R}$, we define the level set S_{λ_0} to be $\{(x, \xi) \in V \mid \tau(x, \xi) = \lambda_0\}$ if $\lambda_0 \neq 0$ and define $S_{\lambda_0} = \{(x, \xi) \in V \mid \tau(x, \xi) = 0, |\xi| = 1\}$ if $\lambda_0 = 0$. We assume that S_{λ_0} is closed in (x, ξ) -space and the null space of $A(x, \xi) - \lambda_0 I$ is one dimensional for $(x, \xi) \in S_{\lambda_0}$. Let $x(t)$, $0 \leq t \leq T$, be the curve determined by

$$\dot{x} = -\partial\tau/\partial\xi, \quad \dot{\xi} = \partial\tau/\partial x,$$

where $(x(0), \xi(0)) = (x_0, \xi_0) \in S_{\lambda_0}$. Then we have

Theorem 4.1. *Given neighborhoods of x_0 and $x_T = x(T)$, N_0 and N_T , there is a $\varphi_\varepsilon \in C_0^\infty(N_0)$ such that $\|\varphi_\varepsilon\|_0 = 1$ and $\|U(t)\varphi_\varepsilon\|_0^{N_T} \geq 1 - \varepsilon$ for all $\varepsilon > 0$.*

Proof. We build $Q(t)$, $0 \leq t \leq T$, as in §3, but here $a(x, \xi, 0)$ is chosen with support in a sufficiently small conic neighborhood of (x_0, ξ_0) that the support of $a(x, \xi, 0)$ in x is contained in N_0 and the support of $a(x, \xi, T)$ in x is contained in N_T . $a(x, \xi, 0)$ is chosen equal 1 on a conic neighborhood of (x_0, ξ_0) .

Let $\varphi(x) = \psi(x)e^{ik\xi_0 \cdot x}$ where $\psi \in C_0^\infty(N_0)$. Since $\varphi \in \mathfrak{D}(L)$ we have

$$\frac{\partial Q(t)U(t)\varphi}{\partial t} - LQ(t)U(t)\varphi = K(t)U(t)\varphi$$

and $K(t)$ is a uniformly (for $0 \leq t \leq T$) bounded operator from $H_{-1}(\mathbf{R}^n)$ to $L^2(\mathbf{R}^n)$. By Duhamel's formula

$$Q(T)U(T)\varphi - U(T)Q(0)\varphi = \int_0^T U(T-s)K(s)U(s)\varphi ds,$$

$$\|Q(T)U(T)\varphi - U(T)Q(0)\varphi\|_0 \leq T \sup_{t \in [0, T]} \|K(t)U(t)\varphi\|_0 \leq C_T \sup_{t \in [0, T]} \|U(t)\varphi\|_{-1}.$$

Hence, using (3.7) with $s = -1$, we conclude

$$\|Q(T)U(T)\varphi - U(T)Q(0)\varphi\|_0 \leq K_T \|\varphi\|_{-1}.$$

As $k \rightarrow \infty$, $\|\varphi\|_{-1} \rightarrow 0$. Since $Q(t)$ was constructed so that $Q(T)f$ vanishes outside a small neighborhood of the support of $a(x, \xi, T)$ in x , we may assume $Q(T)U(T)\varphi = 0$ off N_T . Hence as $k \rightarrow \infty$

$$\|U(T)Q(0)\varphi\|_0^{R^n - N_T} \rightarrow 0.$$

Since $U(t)$ is unitary on L^2 , to prove Theorem 4.1 it suffices to show we can choose ψ so that $\|Q(0)\varphi\|_0 \rightarrow 0$ as $k \rightarrow \infty$. The basic estimate from the symbol calculus of pseudo-differential operators gives

$$e^{-ikx \cdot \xi_0} Q(0)\varphi \rightarrow a(x, \xi_0, 0) P(x, \xi_0) \psi(x)$$

uniformly in x over compact subsets as $k \rightarrow \infty$. Hence, we choose ψ so that $P(x_0, \xi_0)\psi(x_0) \neq 0$ and then $\|Q(0)\varphi\|_0 \rightarrow 0$ as $k \rightarrow \infty$.

Theorem 4.1 has the following corollary.

Corollary 4.2. *Let χ_B denote the characteristic function of $|x| < B$. If $|x(t)| < B$ for all $t > 0$, the norm of the operator $\chi_B U(t)\chi_B$ is one for all $t \geq 0$.*

Corollary 4.2 says no uniform decay estimate is possible if a simple bicharacteristic is trapped for $t > 0$. However, if the bicharacteristic is not a null

bicharacteristic and hypothesis (ii) holds, Corollary 3.4 still implies $\|\chi_B U(t)f\|_0 \rightarrow 0$ as $t \rightarrow \infty$ for all $f \in L^2(\mathbf{R}^n) \ominus H_0$. Our concluding example shows this can fail if there are trapped null bicharacteristics.

Example. We construct a vector field on \mathbf{R}^3 as follows: Let $\psi(r, s) \in C_0^\infty$ with $\psi(r, s) = 0$ for (r, s) outside $\{(r, s) \mid |s| \leq 1, 1 \leq r \leq 2\}$. Assume $0 \leq \psi(r, s) \leq 1$, $\psi(r, 0) = 1$ for $4/3 \leq r \leq 5/3$, and $\psi(s, r) < 1$ for $s \neq 0$. Then we define for $r = (x_1^2 + x_2^2)^{1/2}$, $\mathbf{v}(x_1, x_2, x_3) = \psi(r, x_3)(-x_2/r, x_1/r, 0) + (1 - \psi(r, x_3))(0, 0, 1)$. Note that $\mathbf{v}(\mathbf{x}) = (0, 0, 1)$ for $|\mathbf{x}| > \sqrt{5}$, and $|\mathbf{v}(\mathbf{x})| > 0$ for all \mathbf{x} . Let $\mathbf{x}(t, y)$ be the solution to $\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x})$, $\mathbf{x}(0, y) = y$. $r(t) = (x_1(t)^2 + x_2(t)^2)^{1/2} = (y_1^2 + y_2^2)^{1/2}$ for all t . Hence $x_3(t, y)$ is a nondecreasing function of t with $x_3(t, y) \rightarrow 0$ as $t \rightarrow \infty$ for $y_3 < 0$, $4/3 \leq (y_1^2 + y_2^2)^{1/2} \leq 5/3$, and $x_3(t, y) \rightarrow 0$ as $t \rightarrow -\infty$ for $y_3 > 0$, $4/3 \leq (y_1^2 + y_2^2)^{1/2} \leq 5/3$.

Now let $U(t)$ be the unitary group of $L^2(\mathbf{R}^3)$ given by

$$(U(t)f)(y) = f(\mathbf{x}(t, y))((\partial \mathbf{x} / \partial y)(t, y))^{1/2}.$$

By the preceding remarks, given $f(\mathbf{x})$ supported in $|\mathbf{x}| < \sqrt{5}$, $x_3 < 0$, $4/3 < (x_1^2 + x_2^2)^{1/2} < 5/3$, $\|U(t)f\|_0^{1/5} = \|f\|_0$ for $t > 0$.

One computes $U(t)$ is generated by $L = \mathbf{v} \cdot \nabla + \frac{1}{2} \nabla \cdot \mathbf{v}$. Hence $A(x, \xi) = \tau(x, \xi) = \mathbf{v}(\mathbf{x}) \cdot \xi$. The bicharacteristic through (y, η) has the form $(\mathbf{x}(t, y), \xi(t, y, \eta))$ where $\mathbf{x}(t, y)$ is function defined in the preceding paragraph. If $\mathbf{v}(y) \cdot \eta = 0$, this is a null bicharacteristic. Thus every integral curve of $\mathbf{v}(\mathbf{x})$ is a null bicharacteristic for the proper choice of η .

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